A Practical Introduction to Electricity
Alternating Current

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1 Theory – Alternating Current

As some students are not familiar with the engineering techniques used for describing alternating current (AC) quantities, i. e. the description of harmonic oscillations at a given frequency, a brief theoretical introduction of phasor theory is given here without the use of complex numbers.

1.1 Signal Description

When electrical circuits with *linear components* like resistors, capacitors, and inductors (coils, windings, circuit loops) are operated with a *harmonic* voltage or current source, all voltages and currents in the circuit will be harmonic after a brief transient phase, these signals differing only by their amplitudes and their time shifts. All signals can then be described in function of the time t with the following formula:

$$x(t) = \hat{X}\cos(\omega t + \phi) = \hat{X}\cos(2\pi f t + \phi) \tag{1}$$

The expression (1) contains three parameters which uniquely define the signal:

Amplitude:	$\hat{X} > 0$	with the unit V or A depending on the nature of the signal
Zero Phase Angle:	ϕ	unit: $[\phi] = \text{rad}$ (radian)
Angular Frequency:	$\omega = 2\pi f$	unit: $[\omega] = \text{s}^{-1}$ or rad/s
or **Frequency**:	f	unit: $[f] = \text{Hz}$ (Hertz) or cps (cycles per second)

As all signals show the same frequency, it is sufficient to retain the amplitude and the zero phase angle for each signal:

> Harmonic signals of known (single) frequency can be unambiguously described with two parameters only:
> $$x(t) = \hat{X}\cos(\omega t + \phi) \quad \leftrightarrow \quad (X, \phi) \tag{2}$$

Remarks:

- The parameters $\hat{X}$ and X are positive by definition. It is common not to use the amplitude $\hat{X}$ in the pair on the right side of the mapping (2) but the so called **effective value** (Effektivwert) or **RMS value** (**r**oot **m**ean **s**quare) X of the signal. This latter value corresponding to the amplitude through the following fixed relation:[1]

$$\boxed{X = \frac{\hat{X}}{\sqrt{2}}} \tag{3}$$

> An harmonic AC signal with amplitude $\hat{X}$ delivers an average power to a resistor equal to that of a DC signal with level $X = \hat{X}/\sqrt{2}$.

- $\sin(\omega t + \varphi)$ can be expressed as $\cos(\omega t + \phi)$ with $\phi = \varphi - \pi/2$

[1] This relation holds only for harmonic signals.

- A change in the sign of the signal is described by a change in the zero phase angle of plus or minus π: $-\cos(\omega t + \phi) = \cos(\omega t + \phi \pm \pi)$.

- The derivative of $x(t) = \hat{X}\cos(\omega t + \phi)$ can also be expressed according to the mapping of equation (2):

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \omega\hat{X}\cos(\omega t + \phi + \pi/2) \quad \leftrightarrow \quad (Y, \varphi) = (\omega X, \phi + \pi/2) \tag{4}$$

  The derivation of a harmonic signal stretches the amplitude by ω and augments the zero phase angle by $\pi/2$. Turning the phase by $+\pi/2$ is equivalent to an anticipated time shift by a quarter period.

- The pair (X, ϕ) in equation (2) can be interpreted as the *polar coordinates* of a point in the xy-plane, the *cartesian coordinates* being $(X\cos\phi, X\sin\phi)$.
  This point is called **phasor** (Zeiger, Festzeiger) and can also be interpreted as a vector pointing from the origin of the coordinate system to the point itself.

  Remark: The classical description of AC signals uses complex numbers but we shall do without this concept for the time being.

- The mapping defined by equation (2) is *linear*. That signifies first, that multiplying a signal by a constant value means also multiplying the effective value by the same constant (*homogeneity property*), and secondly, adding different harmonic signals with the same frequency in the time domain can be mapped to a (cartesian) vectorial sum of the different phasors belonging to the signals (*additivity property*).

$$\begin{aligned}
y(t) &= \alpha x(t) = \alpha\hat{X}\cos(\omega t + \phi) \quad \leftrightarrow \\
(Y\cos\phi, Y\sin\phi) &= (\alpha X\cos\phi, \alpha X\sin\phi) \qquad \text{(Homogeneity)} \\
x(t) &= x_1(t) + x_2(t) = \hat{X}_1\cos(\omega t + \phi_1) + \hat{X}_2\cos(\omega t + \phi_2) \quad \leftrightarrow \\
(X\cos\phi, X\sin\phi) &= (X_1\cos\phi_1 + X_2\cos\phi_2, X_1\sin\phi_1 + X_2\sin\phi_2) \qquad \text{(Additivity)}
\end{aligned}$$

  Geometrically this corresponds to a vectorial addition of the phasors. Adding phasors is in fact much easier to compute than trigonometric time signals!

1.2 AC Behavior of R, C and L Components

For the next section we shall make use of the following general definitions:

$$\text{Phasor for } \; u(t) = \hat{U}\cos(\omega t + \varphi_u) \;\; \leftrightarrow \;\; (U, \varphi_u) \tag{5}$$

$$\text{Phasor for } \; i(t) = \hat{I}\cos(\omega t + \varphi_i) \;\; \leftrightarrow \;\; (I, \varphi_i) \tag{6}$$

$$\textbf{Impedance}: \quad Z = \frac{U}{I} = \frac{\hat{U}}{\hat{I}} \tag{7}$$

$$\textbf{Admittance}: \quad Y = \frac{1}{Z} = \frac{I}{U} = \frac{\hat{I}}{\hat{U}} \tag{8}$$

$$\textbf{Phase Angle}: \quad \varphi = \varphi_u - \varphi_i \tag{9}$$

Resistance

In linear resistors the voltage is proportional to the current at any time:[2]

$$\boxed{\frac{u(t)}{i(t)} = R}$$

In this case the phasor of the voltage is parallel to the one of the current (see figure 1). The ratio between U and I is equal to the resistance and the phase angle φ equals zero.

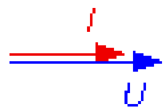

Figure 1: Phasor Pair of Voltage and Current for a Linear Resistor
The direction of the two phasors is parallel independently of their common orientation.

$$\boxed{\begin{aligned} &Z = R \\ &Y = G = \frac{1}{R} \\ &\varphi = 0 \quad \text{as} \quad \varphi_u = \varphi_i \end{aligned}} \tag{10}$$

Capacitance

In linear capacitors – capacitors where the capacitance C does not depend on the voltage applied – the charge $q(t)$ is proportional to the applied voltage $u(t)$ at any time.[3] This ratio is the capacitance C:

$$\boxed{C = \frac{q(t)}{u(t)}}$$

According to the *law of conservation of charge* the current in linear capacitors is proportional to the rate of change of the charge and therefore to the rate of change of the voltage:

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} \tag{11}$$

Using the linearity of the mapping (2) and the relation (4) for derivatives it is obvious that the following holds:

$$\boxed{i(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} \quad \leftrightarrow \quad (I, \varphi_i) = (\omega C U, \varphi_u + \pi/2)} \tag{12}$$

In this case the phasor of the voltage is at right angle behind the one of the current and the following holds:

$$\boxed{\begin{aligned} &Z = \frac{1}{\omega C} \\ &Y = \omega C \\ &\varphi = -\frac{\pi}{2} \quad \text{as} \quad \varphi_i = \varphi_u + \frac{\pi}{2} \end{aligned}} \tag{13}$$

[2] These behave according to Ohm's law, where the resistance R does not depend on the current or the voltage.
[3] In a capacitor both electrodes carry the same amount of charge the only difference being the sign.

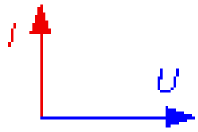

Figure 2: Phasor Pair of Voltage and Current for a Linear Capacitor
There is a right angle between the two phasors. The harmonic current is leading (in a mathematically positive sense of rotation, that means counter-clockwise) for a time interval of a quarter period compared to the voltage.

Self Inductance

In linear inductors – Coils or windings where the inductance L does not depend on the current – the magnetic flux $\psi(t)$ is proportional to the flowing current $i(t)$ at any time. This ratio is the inductance or self inductance L:

$$\boxed{L = \frac{\psi(t)}{i(t)}}$$

According to the *law of induction* the induced voltage in linear inductors is proportional to the rate of change of the magnetic flux and therefore to the *rate of change of the current*:

$$u(t) = \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \tag{14}$$

Using the linearity of the mapping (2) and the relation (4) for derivatives it is obvious that the following holds:

$$\boxed{u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \quad \leftrightarrow \quad (U, \varphi_u) = (\omega L I, \varphi_i + \pi/2)} \tag{15}$$

In this case the phasor of the current is at right angle behind the one of the voltage and

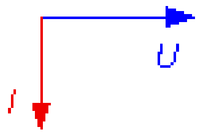

Figure 3: Phasor Pair of Voltage and Current for a Linear Inductor
There is a right angle between the two phasors. The harmonic voltage is leading for a time interval of a quarter period compared to the current.

the following holds:

$$\boxed{\begin{aligned} &Z = \omega L \\ &Y = \frac{1}{\omega L} \\ &\varphi = +\frac{\pi}{2} \quad \text{as} \quad \varphi_u = \varphi_i + \frac{\pi}{2} \end{aligned}} \tag{16}$$

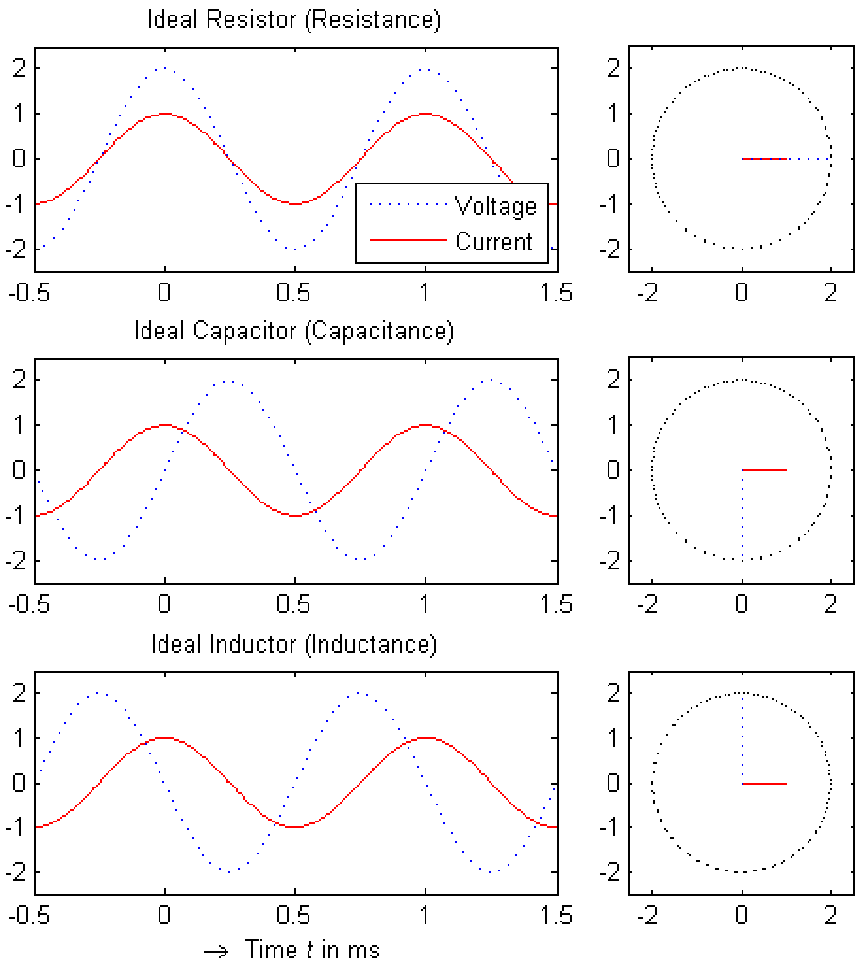

Figure 4: Voltage and Current in the Time Domain with their Corresponding Phasors

Mutual Inductance

In inductors the phenomenon of induction arises independently of the origin of the magnetic flux. In the case of self inductance the flux is generated by the own current of the inductor. If the flux is produced by some external source it will nevertheless induce a voltage in any circuit exposed to that flux. Basically both phenomenons, self and mutual inductance will be present and the parts of the resulting flux cannot anymore be attributed to their origin.
The mutual inductance can be defined the same way as for the self inductance:

$$L_{12} = \frac{\psi_{12}(t)}{i_2(t)}$$

The first index referring to the location where the flux is interacting with the circuit and the second referring to the source of the magnetic field, that is the current $i_2(t)$.

When two circuits are magnetically coupled, they will influence each other even when they are not electrically connected. So, if there is a mutual inductance L_{12} from one circuit to the other, there will be one L_{21} acting in the other direction. For linear circuits it appears that the two mutual inductances will have exactly the same value:[4]

$$\boxed{L_{12} = L_{21} = M} \tag{17}$$

In the presence of self and mutual inductance the induced voltages in both circuits will look as follows:

$$\begin{aligned} u_1(t) &= L_1 \frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + M \frac{\mathrm{d}i_2(t)}{\mathrm{d}t} \\ u_2(t) &= M \frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2(t)}{\mathrm{d}t} \end{aligned} \tag{18}$$

The voltage $u_1(t)$ being the induced voltage in the first and $u_2(t)$ the one in the second circuit or winding. As the external flux can have two different directions and so induce positive or negative voltages depending on the orientation of the objects, the mutual inductance M will carry a sign depending on this orientation and on the reference direction of the external current.

The following important relation between the self inductances L_1 and L_2 of two coupled circuits and their mutual inductance M holds:

$$\boxed{|M| \leq \sqrt{L_1 L_2}} \tag{19}$$

For the mutual inductance the phasors behave almost exactly as the case the self inductance:

$$\begin{aligned} Z = &\quad \omega|M| \\ Y = &\quad \frac{1}{\omega|M|} \\ \varphi = \pm\frac{\pi}{2}\text{as} &\quad \varphi_u = \varphi_i \pm \frac{\pi}{2} \end{aligned} \tag{20}$$

The negative sign being for the case of a negative value of M.

[4] This is the reason why some authors use the symbol M for this variable as we shall do here.

1.3 AC Power Concept

Instantaneous Power

Instantaneous power (Momentanleistung) is defined as the actual energy flow entering a two-terminal device (Zweipol).[5] This power can be expressed as follows (assuming for simplicity that the zero phase angle of the current φ_i is zero, i. e. $\varphi = \varphi_u$):

$$\begin{aligned} p(t) &= u(t) \cdot i(t) = \hat{U}\cos(\omega t + \varphi) \cdot \hat{I}\cos(\omega t) \\ &= UI\cos\varphi + UI \cdot \cos(2\omega t + \varphi) \qquad (21) \\ &= UI\cos\varphi + UI\cos\varphi \cdot \cos\big(2(\omega t + \varphi)\big) + UI\sin\varphi \cdot \sin\big(2(\omega t + \varphi)\big) \qquad (22) \end{aligned}$$

In this transformation we replaced the amplitudes $\hat{U}$ and $\hat{I}$ of the voltage and the current with their effective values U and I according to expression (3).

Apparent Power

With the definition of the so called apparent power

$$\boxed{S = UI} \qquad (23)$$

equation (21) can be written as

$$p(t) = S\cos\varphi + S \cdot \cos(2\omega t + \varphi).$$

This function is shown in the center graph of figure 5.

Real Power

Real power is the average of the energy flow into the two-terminal device:[6]

$$P = \frac{1}{T}\int_T p(t)\,\mathrm{d}t$$

From equation (21) and from the center graph of figure 5 it is obvious that the following relation holds

$$\boxed{P = S\cos\varphi} \qquad (24)$$

The ratio between the real and the apparent power is called **power ratio**: $\lambda = \frac{P}{S} = \cos\varphi$.

Reactive Power

Reactive power is the amplitude of the energy flow periodically exchanged between the two-terminal device and the supplying source. From equation (22) and from the lower graph of figure 5 it is obvious that the following relation holds

$$\boxed{Q = S\sin\varphi} \qquad (25)$$

The sign of the reactive power will be same as the one of the phase angle. A positive value signifies that the device shows *inductive behavior*, a negative one that it shows *capacitive behavior*.

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[5] We assume here, that the reference arrows (Bezugsrichtungspfeile) for the voltage and the current are directed so, that the one for the energy flow or power shows into the two-terminal.

[6] The integration has to be carried out over a full period $T = 1/f$ of the power signal.

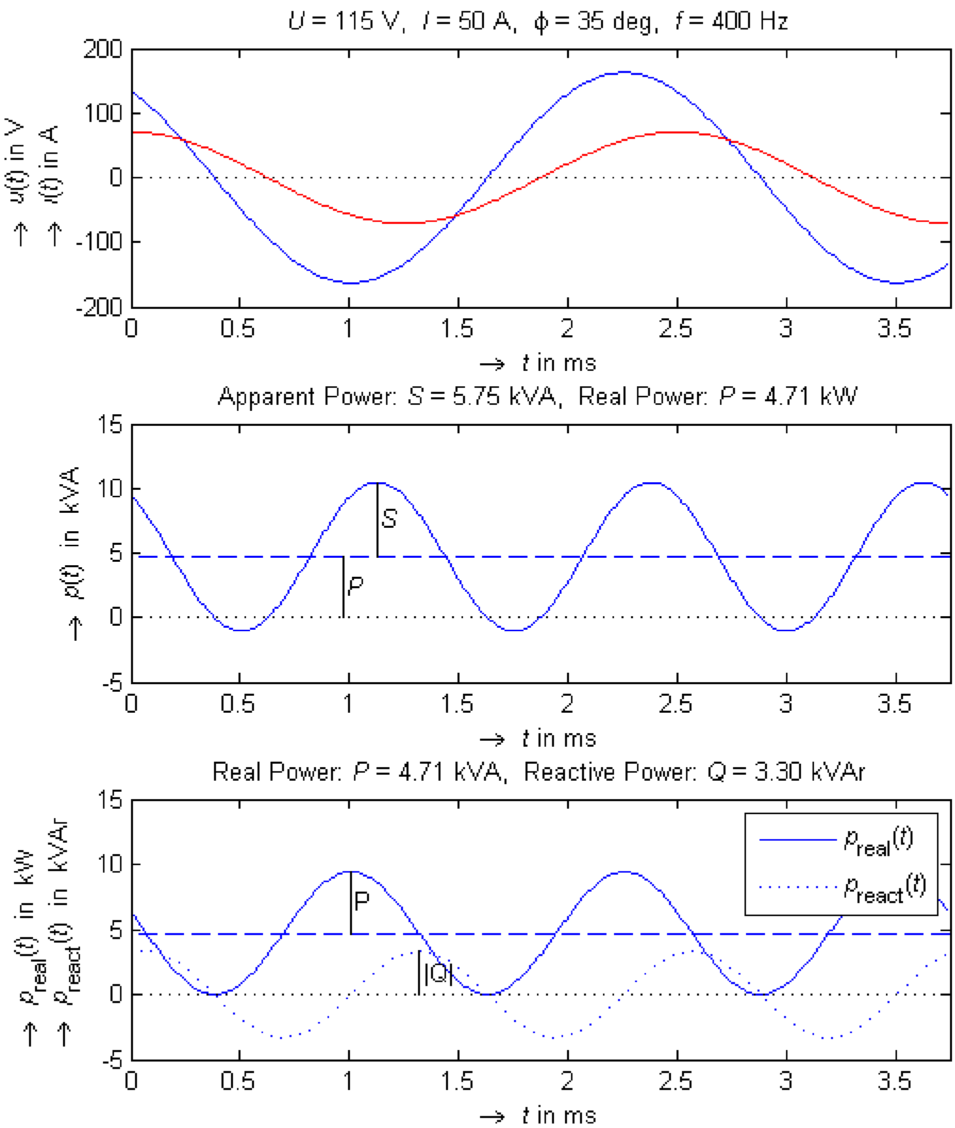

Figure 5: Instantaneous Power Decomposed in Real and Reactive Power Parts:
$p(t) = p_{real}(t) + p_{react}(t)$
In the upper graph the voltage and the current of a typical 115 V / 400 Hz single-phase AC-system are shown. The center graph shows the corresponding instantaneous power. Note the doubled frequency, the average level, and the amplitude of this signal. During the negative phase of the instantaneous power the energy flow is reversed, that means energy flows out of the two-terminal device. The lower graph shows the two components of the decomposed instantaneous power. The amplitude of the real part corresponds to its mean level, while the reactive part oscillates around zero. It is obvious that the reactive power is periodically exchanged between the two-terminal device and the source. Reactive power does not get lost but can lead to losses when transported over longer lines (Leitungen).

Figure 6: A possibility to remember the differences...

2 Practical Experiments

The purpose of the following experiments is to introduce the techniques used for computing and measuring alternating current properties such as capacitance, self and mutual inductance found in capacitors and coils. A brief theoretical and specific introduction to the considered linear elements is given at the beginning of each unit despite of the introductory chapter. The practical work should be properly documented, the documentation being an important aspect of the issue.

2.1 Capacitor and Capacitance

Learning Objectives

The assignment of this unit is to measure the capacitance of a capacitor using AC current and to investigate its behavior in combination with a resistor.
The students shall get to know the following relations and methods:

- constitutive law: relation between voltage and current in an ideal capacitor
- phasor description of the behavior of a capacitor and a resistor connected in series
- measurement and calculation techniques for determining the amplitude ratio and the phase angle between current and voltage
- method for measuring the capacitance of linear capacitors

Theoretical Introduction

According the the relation (12) the following fundamental rule holds *independently of the signal waveform*:

$$i(t) = C\frac{\mathrm{d}u_C(t)}{\mathrm{d}t} \tag{26}$$

Setting $u_C(t) = \hat{U}_C \cos \omega t$ for the harmonic voltage over the capacitor, we find out:[7]

$$i(t) = -C\omega\hat{U}_C \sin \omega t = \omega C\hat{U}_C \cos(\omega t + \pi/2) \quad \leftrightarrow \quad (I, \varphi_i) = (\omega C U_C, \pi/2) \tag{27}$$

Using the effective value of the current expressed in function of the voltage from equation (27) we get, in accordance with equation (13):

$$\begin{aligned} I &= \omega C U_C \\ C &= \frac{I}{\omega U_C} = \frac{1}{\omega}\frac{\hat{I}}{\hat{U}_C} \end{aligned}$$

As the phase angle φ is $-\pi/2$ for an ideal capacitor the reactive power is negative:

$$\begin{aligned} Q &= S \sin\varphi = -U_C I = -\omega C U_C^2 \\ S &= -Q \end{aligned}$$

Practical Part

For measuring the capacitance of a capacitor the schematic according to figure 7 can be used. The amplitudes of $u_C(t)$ and the current $i(t) = u_R(t)/R$ can be measured with a scope (Oszilloskop). Eventually the capacitance can be calculated with

$$C = \frac{I}{\omega U_C} = \frac{1}{2\pi f}\frac{I_{pp}}{U_{Cpp}} = \frac{1}{2\pi f}\frac{U_{Rpp}}{R\,U_{Cpp}}$$

The index "*pp*" meaning that the values can be measured peak-to-peak for better precision.

Remark: Basically the amplitudes of $u_C(t)$ and $i(t)$ could be measured directly and more precisely as effective values with two digital multimeters (DMM) if the latter *work properly up to the frequency used.*

The voltages $u_R(t)$ and $u_C(t)$ cannot be measured simultaneously with a scope. It is however possible to calculate the RMS value of $u_R(t)$ using the according values of the measured voltages $u(t)$ and $u_C(t)$. The geometrical relations in the phasor diagram shown in figure 8 lead to the following equation which holds for peak-to-peak values too:

$$U^2 = U_C^2 + U_R^2 \quad \rightarrow \quad U_R = \sqrt{U^2 - U_C^2}$$

Evaluation

Investigate the following questions:

- Compare the measurement of the capacitance done with the scope to the one done with two DMM. Which one is more reliable? Compare the results with the nominal value of the capacitor.

[7] Without loss of generality we can set the zero phase angle of the voltage to zero: $\varphi_u = 0$.

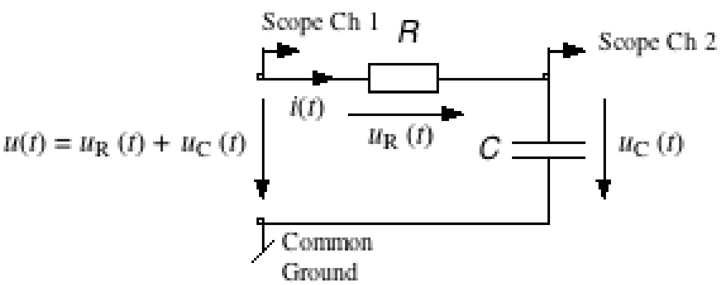

Figure 7: Schematic for Measuring the Capacitance of a Capacitor with AC Excitation
As a scope measures only voltages the current cannot be measured directly. Therefore a resistor must be introduced in the circuit. The frequency f of the AC source and the resistance R have to be selected properly in order to get useable signals.
The scope measures the voltages referring to the common ground, so that $u_R(t)$ cannot be recorded together with $u_C(t)$. This could be achieved by recording the difference between $u(t)$ and $u_C(t)$ with the scope (Chanel 1 minus Chanel 2).

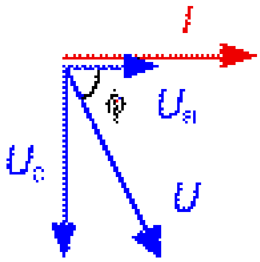

Figure 8: Phasor Diagram of the Measurement Circuit

- Does the capacitance change with frequency? If so, how? Remark: Don't go below 100 Hz or beyond 100 kHz.
- Is the phase angle of the capacitor between $u_C(t)$ and $i(t)$ really $-\pi/2$ and does the voltage lag behind the current by a quarter of the period? How would you explain a phase angle having a magnitude less than $\pi/2$?
- Explain why the magnitude of the reactive power increases with frequency.

2.2 Inductor and Inductance

Learning Objectives

The assignment of this unit is to measure the inductance of a coil using AC current and to investigate its behavior in combination with a resistor.
The students shall get to know the following relations and methods:

- constitutive law: relation between voltage and current in an ideal coil
- phasor description of the behavior of a inductor and a resistor connected in series
- measurement and calculation techniques for determining the amplitude ratio and the phase angle between current and voltage
- method for measuring the inductance of linear inductors

Theoretical Introduction

According the the relation (15) the following fundamental rule holds *independently of the signal waveform*:

$$u_L(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \tag{28}$$

Setting $i(t) = \hat{I}\cos\omega t$ for the harmonic current, we find out:[8]

$$u_L(t) = -L\omega\hat{I}\sin\omega t = \omega L\hat{I}\cos(\omega t + \pi/2) \quad \leftrightarrow \quad (U_L, \varphi_u) = (\omega LI,\, \pi/2) \tag{29}$$

Using the effective value of the voltage expressed in function of the current from equation (29) we get, in accordance with equation (16):

$$\begin{aligned} U_L &= \omega LI \\ L &= \frac{U_L}{\omega I} = \frac{1}{\omega}\frac{\hat{U}_L}{\hat{I}} \end{aligned}$$

As the phase angle φ is $\pi/2$ for an ideal inductor the reactive power is positive:

$$\begin{aligned} Q &= S\,\sin\varphi = U_L I = \omega LI^2 \\ S &= Q \end{aligned}$$

Practical Part

For measuring the inductance of a linear coil the schematic according to figure 9 can be used.[9] The amplitudes of $u(t)$, $u_S(t)$ and the current $i(t) = u_m(t)/R_m$ can be measured with a scope (Oszilloskop). Eventually the inductance can be calculated with

$$L = \frac{U_L}{\omega I} = \frac{1}{2\pi f}\frac{U_{Lpp}}{I_{pp}} = \frac{1}{2\pi f}\frac{U_{Lpp}\,R_m}{U_{mpp}}$$

The index "pp" meaning that the values can be measured peak-to-peak for better precision.

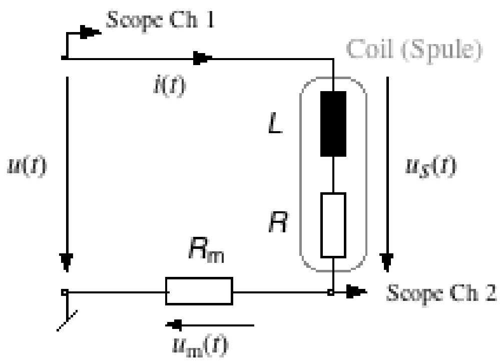

Figure 9: Schematic for Measuring the Inductance of a Inductor with AC Excitation
As a scope measures only voltages the current cannot be measured directly. Therefore a resistor must be introduced in the circuit. The frequency f of the AC source and the resistance R_m have to be selected properly in order to get useable signals.
The scope measures the voltages referring to the common ground, so that $u_m(t)$ cannot be recorded together with $u_S(t)$. This could be achieved by recording the difference between $u(t)$ and $u_m(t)$ with the scope (Chanel 1 minus Chanel 2).

Remark: Basically the amplitudes of $u(t)$, $u_S(t)$ and $i(t)$ could be measured directly and more precisely as effective values with digital multimeters (DMM) if the latter *work properly up to the frequency used*.

Unfortunately the voltage $u_L(t)$ cannot be measured directly because of the resistance of the inductor: $u_S(t) = u_L(t) + u_R(t)$. It is however possible to calculate the RMS values of $u_S(t)$ and $u_L(t)$ using the according values of the measured voltages $u(t)$ and $u_m(t)$. The geometrical relations in the phasor diagram shown in figure 10 lead to the following set of equations which hold for peak-to-peak values too:

$$U_m = R_m I \qquad \rightarrow \qquad U_R = RI = \frac{R}{R_m} U_m$$
$$U^2 = U_L^2 + (U_R + U_m)^2 \qquad \rightarrow \qquad U_L = \sqrt{U^2 - (U_R + U_m)^2}$$

The value of the resistance R of the inductor has of course to be measured separately using a DMM or DC techniques.

The inductance can then be easily calculated by

$$U_L = \omega L I \quad \rightarrow \quad L = \frac{1}{\omega}\frac{U_L}{I} = \frac{1}{2\pi f}\frac{U_L}{U_m} R_m$$

Evaluation

Investigate the following questions:

- Compare the measurement of the self inductance done with the scope to the one done with two DMM. Which one is more reliable?

[8] Without loss of generality we can set the zero phase angle of the current to zero: $\varphi_i = 0$.
[9] The behavior of a linear coil in a restricted frequency range can be modeled by an ideal inductor in series to an ideal resistor.

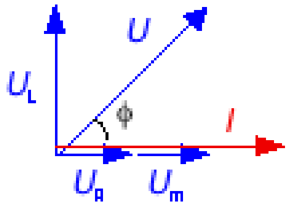

Figure 10: Phasor Diagram of the Measurement Circuit

- Does the inductance change with frequency? If so, how? Remark: Don't go below 100 Hz or beyond 100 kHz.
- Is the phase angle of the inductor between $u_L(t)$ and $i(t)$ really $\pi/2$ and does the current lag behind the voltage by a quarter of the period?
- The inductive quality of a coil can be expressed with the following **quality factor** Q_L (Spulengüte):[10]

  $$Q_L = \frac{Q}{P} = \frac{\omega L}{R}$$

  What does the phase angle between $u_S(t)$ and the current $i(t)$ have to do with Q_L? How does Q_L depend on the frequency?
- Explain why the reactive power increases with frequency.

[10] This factor should not be confused with the reactive power even if its symbol is almost the same! Note the index $_L$.

2.3 Resonance and Reactive Power Compensation

Learning Objectives

Energy can be stored in capacitors together with charge or in inductors coupled to magnetic flux. The purpose of this unit is to investigate the interactions of these two different kinds of energy storage.
The students shall get to know the following phenomenons and concepts:

- the resonance in an electrical circuit
- reactive power compensation

Theoretical Introduction

Most energy using devices as electrical motors, drives and actuators are inductive, leading to unwanted losses as reactive power leads to unnecessary high currents or voltages. To understand why, it is necessary to see what happens in the phenomenon of *resonance*. For this we shall look at the following circuit:

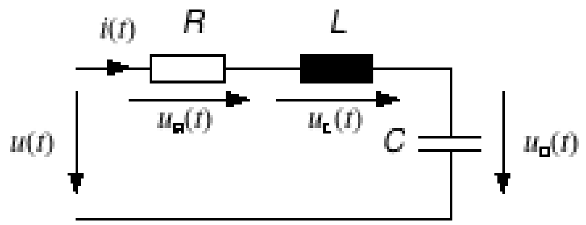

Figure 11: Schematic of a Series Resonant Circuit

According to the mesh law the equation for the voltages becomes

$$u(t) = u_R(t) + u_L(t) + u_C(t).$$

This equation can certainly not be transferred directly to the effective values of the voltages as the phase angles are not synchronous. Therefore we shall look at the phasor diagram of the circuit shown in figure 12. The phasor (U, φ) of the total voltage can be calculated by vectorial addition of the voltage phasors starting from the current common to all elements. Together with the theorem of Pythagoras the diagram yields:

$$U = \sqrt{U_R^2 + (U_L - U_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, I \quad (30)$$

$$\varphi = \arctan \frac{U_L - U_C}{U_R} = \arctan \frac{\omega L - \frac{1}{\omega C}}{R} \quad (31)$$

Under the condition $\omega L = \frac{1}{\omega C}$ the equations (30) and (31) reduce to $U/I = R$ and $\varphi = 0$. That means that the impedance $Z = U/I$ reaches a minimum and that the time dependent

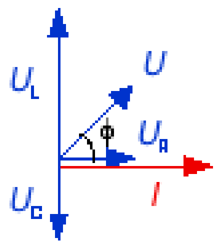

Figure 12: Phasor Diagram of the RLC-Serial Circuit

voltage $u(t)$ and the current $i(t)$ at the connectors of the circuit are synchronous. This condition leads to the so called phenomenon of **resonance**. The (angular) frequency for which

$$\boxed{\omega_r = 1/\sqrt{L\,C}} \tag{32}$$

holds is called **resonance frequency**.

Remarks:

- At resonance the apparent power is equal to the real power at the connectors of the circuit: $S = P$. This is equivalent to $\varphi = 0$. There will still be reactive power in the circuit, but it will not be exchanged through the connectors but between the capacitor and the inductor within the circuit itself. The voltages over these two components can be a lot higher than the one seen at the entrance of the circuit.

- The result of equation (32) is also true for a parallel resonance circuit where the three elements are connected in parallel. In this case the current inside the circuit can be a lot higher than the one seen at the circuit connectors.

- For a given voltage U and a definite amount of real power P, the lowest possible current flows at resonance: $I = P/U$ (because of $S = UI \geq P$). Under this condition the current needed to provide the required real power is the smallest possible and this leads to less power line losses. The choice for the capacitance where the voltages over the inductor will exactly compensate the one over the capacitor for a given frequency is therefore:

  $$C = \frac{1}{\omega^2 L}$$

  This technique is called **reactive power compensation**.

Practical Part

Build a series resonance circuit according to figure 13. Choose a capacitor so as to reach a resonance frequency somewhere between 2 and 10 kHz. The resistor R_m is necessary to measure the current with the scope. The total resistance $R_{tot} = R + R_m$ of the circuit should fulfill the following condition:

$$R_{tot} < \frac{1}{5}\sqrt{\frac{L}{C}}$$

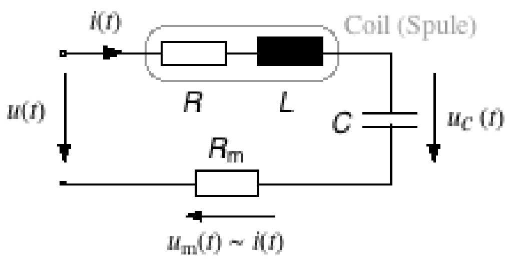

Figure 13: Schematic of the RLC-Serial Resonance Circuit

Calculate the capacitance needed to compensate the reactive power of the circuit with a given coil at a frequency of 400 Hz.

The reactive power can be compensated through connecting the capacitor parallel to the coil as shown in figure 14:

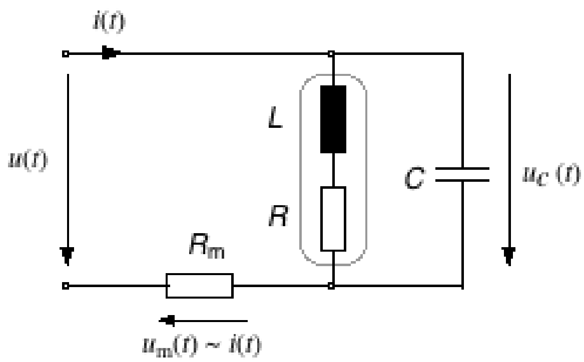

Figure 14: Schematic of the Circuit for Reactive Power Compensation

For this circuit the resonance frequency is given by the following equation:

$$\omega_r = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2} \quad \text{under the condition} \quad \frac{L}{C} > R^2$$

If $R^2 \ll L/C$ the formula for the resonance frequency according to equation (32) is still a good approximation.

Evaluation

Investigate the following questions:

- Compare the calculated resonance frequency of the serial circuit with the one obtained by tuning the frequency until resonance is attained. How sensitive is the resonance condition to changes in frequency?

- How does the resonance voltage $u_C(t)$ over the capacitor depend on the overall resistance R of the serial circuit? What happens if the resistance gets smaller/bigger? Check the following identity at resonance:

$$\frac{U_C}{U} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

  For which resistance does the ratio between U_C and U reach the value of $1/\sqrt{2} \approx 0.7$?

- How well can the compensation of reactive power be realized at 400 Hz with the available capacitors? In which range of values can the capacitance be chosen when the reactive power compensation has to attain a power ratio of $\lambda > 0.95$ at a given frequency, say 400 Hz?

- What is different compared to the serial circuit for a circuit where the coil with R and L is parallel to the capacitor C as in figure 14? How well does the condition according to equation (32) hold?

2.4 Mutual Inductance and Transformer

Learning Objectives

The purpose of this unit is to measure the mutual inductance and understanding the general mode of functioning of an electrical transducer or transformer.[11]
The students shall get to know the following methods and concepts:

- measuring the mutual inductance of magnetically connected circuits
- measuring the transmission factors of the voltage and current conversions

Theoretical Introduction

A model of a transducer or transformer is shown in figure 15

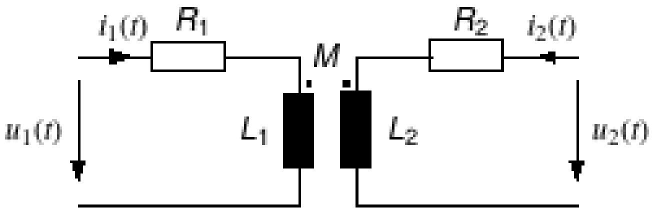

Figure 15: Simple Model of a Transformer/Transducer
Note the two dots (*Kopplungspunkte*) next to the symbols of the inductances. These indicate the polarity of the voltages: When the chosen reference directions of the currents both show to or away from the dots, the mutual inductance is positive and negative in the other case. In this example the sign is positive.

For the voltages in function of the currents and their rate of change the following general relation holds (see equation (18) together with the voltage drop at the resistors):

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}i_1}{\mathrm{d}t} \\ \frac{\mathrm{d}i_2}{\mathrm{d}t} \end{pmatrix} + \begin{pmatrix} R_1 & \\ & R_2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

If the secondary side is open, that means when $i_{20}(t) = 0$, these equations reduce to the simple set of equations

$$\begin{aligned} u_{10}(t) &= L_1 \frac{\mathrm{d}i_{10}(t)}{\mathrm{d}t} + R_1 i_{10}(t) \\ u_{20}(t) &= M \frac{\mathrm{d}i_{10}(t)}{\mathrm{d}t} \end{aligned} \tag{33}$$

From equation (33) using the corresponding phasor diagram it is easy to calculate the mutual inductance M:

$$|M| = \frac{U_{20}}{\omega\, I_{10}}$$

The same procedure with the primary side as open circuit, should theoretically give the same value for the mutual inductance according to equation (17). The sign of M can be determined according to the note of figure 15.

[11] **Tranformers** transform AC voltages and currents for transporting energy at a *fixed frequency* while **transducers** usually transport low power information signals over a *wide range of frequencies*.

Practical Part

All the following measurements can be done with two digital multimeters (DMM) as long as the frequency ist not outside their specified application range.[12] For higher frequencies it is ultimativly necessary to use a scope.

- Begin with measuring the resistors R_1 and R_2 of the windings of the supplied transducer. This can be done with a DMM or using DC.
- Measure and calculate with the described technique and the method of section 2.2 the inductances L_1, L_2 and the two values of M measured from both sides of the transducer.
- Measure the **transformation ratio** $u_{u_{12}}$ of the two voltages (*Spannungsübersetzungsverhältnis*) and $u_{i_{12}}$ of the two currents (*Stromübersetzungsverhältnis*) when the secondary side is connected to the nominal resistive load (see data sheet of the transducer or choose a resistance between 5 and 50 Ω):

$$
\begin{aligned}
u_{u_{12}} &= \frac{U_1}{U_2} \\
u_{i_{12}} &= \frac{I_1}{I_2}
\end{aligned}
$$

Evaluation

Investigate the following questions:

- Compare the two measured values of the mutual inductances M. Guess if the difference could be explained by uncertainties in measurement.
- Is the mutual inductance (take the average value) smaller than the geometrical mean of the inductances according to equation (19)? Determine the **coupling coefficient** (*Kopplungsfaktor*):

$$k_{12} = \frac{M}{\sqrt{L_1\,L_2}} \qquad (|k_{12}| \leq 1)$$

- Check if the transformation ratio of the currents is the reciprocal value of the transformation ratio of the voltages. How different are these ratios from the ratio of the number of windings of the two coils?

[12] Using true RMS instruments is important if the signals are not exactly harmonic. The latter could be checked with a scope.